QUANTIZATION OF THE SPACE OF CONFORMAL BLOCKS.

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Abstract. We consider the discrete Knizhnik-Zamolodchikov connection (qKZ) associated to $gl(N)$, defined in terms of rational R-matrices. We prove that under certain resonance conditions, the qKZ connection has a non-trivial invariant subbundle which we call the subbundle of quantized conformal blocks. The subbundle is given explicitly by algebraic equations in terms of the Yangian $Y(gl(N))$ action. The subbundle is a deformation of the subbundle of conformal blocks in CFT. The proof is based on an identity in the algebra with two generators x, y and defining relation $xy = yx + yy$.

1. Introduction

Conformal Field Theory (CFT) associates a finite dimensional vector space, called the space of conformal blocks, to each Riemann surface with marked points and certain additional data (local coordinates, representations). The vector spaces corresponding to different complex structures or different positions of the points are locally (projectively) identified by a projectively flat connection.

A Wess-Zumino-Witten (WZW) model is labeled by a simple Lie algebra $\mathfrak{g}$ and a positive integer c called level. The space of conformal blocks is defined in terms of the representation theory of the affine Kac-Moody algebra $\widehat{L\mathfrak{g}}$, which is a central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$, see [K]. For each irreducible highest weight $\mathfrak{g}$-module L, we have a canonically defined corresponding irreducible highest weight $\widehat{L\mathfrak{g}}$ module of level c, denoted $\widehat{L}$, see [K]. Then, the space of conformal blocks associated to a Riemann surface Σ, n distinct points with a choice of local holomorphic parameters around them and n irreducible finite dimensional $\mathfrak{g}$ modules $L_1, \dots, L_n$ (such that $\widehat{L}_i$ are integrable) is the space $H^0(\mathfrak{g}(\Sigma), (\widehat{L}_1 \otimes \dots \otimes \widehat{L}_n)^*)$ of linear forms on $\widehat{L}_1 \otimes \dots \otimes \widehat{L}_n$ invariant under the action of the Lie algebra $\mathfrak{g}(\Sigma)$ of meromorphic $\mathfrak{g}$-valued functions on Σ, holomorphic outside of the marked points. The action of $\mathfrak{g}(\Sigma)$ is defined through the Laurent expansion at the poles. Varying the data makes the spaces of conformal blocks into a holomorphic vector bundle with a projectively flat connection given by the Sugawara-Segal construction.

An explicit description is known on the Riemann sphere $\mathbb{P}^1$. Namely, the space of conformal blocks on $\mathbb{P}^1$ with $n+1$ distinct points $z_1, \dots, z_n \in \mathbb{C} \subset \mathbb{P}^1$, $z_{n+1} = \infty$, associated to $n+1$ irreducible $\mathfrak{g}$ modules $L_1, \dots, L_{n+1}$ with highest weights $\lambda_1, \dots, \lambda_{n+1}$, is identified with a subspace of $(L_1 \otimes \dots \otimes L_n)^{sing}_{\lambda}$, where $(L_1 \otimes \dots \otimes L_n)^{sing}_{\lambda} \subset L_1 \otimes \dots \otimes L_n$

is the weight subspace of singular vectors of total weight λ, $\lambda = -w\lambda_{n+1}$ and w is the longest element of the Weyl group. More precisely the space of conformal blocks can be described as follows. Let θ be the highest root and let the scalar product be normalized by $(\theta, \theta) = 2$.

If the resonance condition,

$$(\theta, \lambda) - c + k - 1 = 0, \tag{1}$$

holds for some $k \in \mathbb{N}$, then the space of conformal blocks is identified with subspace

$$W_{\lambda_{n+1}}(z) = \{m \in (L_1 \otimes \ldots \otimes L_n)_\lambda^{sing} \mid (E(z))^k m = 0\},$$

otherwise the space of conformal blocks is identified with the weight space of singular vectors, $W_{\lambda_{n+1}}(z) = (L_1 \otimes \ldots \otimes L_n)_\lambda^{sing}$. Here

$$E(z) = \sum_{i=0}^{n} z_i e_\theta^{(j)},$$

and $e_\theta^{(j)}$ denotes the element $e_\theta \in \mathfrak{g}$ acting on the j-th factor, see [KT], [FSV1].

The subspaces $W_{\lambda_{n+1}}(z)$ associated to different sets of marked points $z_1, \ldots, z_n \in \mathbb{C}$ form a holomorphic subbundle of the trivial vector bundle over the configuration space $\mathbb{C}^{[n]} = \{z \in \mathbb{C}^n \mid z_i \neq z_j,\, i, j = 1, \ldots, n,\, i < j\}$ with fiber $L_1 \otimes \ldots \otimes L_n$. There is a flat connection on the bundle $\mathbb{C}^{[n]} \times (L_1 \otimes \ldots \otimes L_n)$ preserving the subbundle of conformal blocks. Its horizontal sections $\Psi(z)$ obey the Knizhnik-Zamolodchikov equation,

$$\partial_{z_i} \Psi(z) = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \Psi(z), \qquad i = 1, \ldots, n, \tag{2}$$

$\kappa = c + h^\vee$, where Ω_{ij} is the Casimir operator acting on the i-th and j-th factors and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Let $\mathfrak{g} = gl(N)$. The quantized Knizhnik-Zamolodchikov (qKZ) equation is a holonomic system of difference equations for a function $\Psi(z)$ with values in $L_1 \otimes \ldots \otimes L_n$,

$$\Psi(z_1, \ldots, z_i + p, \ldots, z_n) = R_{m,m-1}(z_m - z_{m-1} + p) \ldots R_{m,1}(z_m - z_1 + p) \times$$
$$\times R_{m,n}(z_m - z_n) \ldots R_{m,m+1}(z_m - z_{m+1}) \Psi(z),$$

$i = 1, \ldots, n$, where $R_{i,j}(x)$ is the rational R-matrix $R_{L_{\lambda_i} L_{\lambda_j}}(x)$ acting in i-th and j-th factors and $p \in \mathbb{C}$ is a parameter, see [FR].

The qKZ equation defines the discrete Knizhnik-Zamolodchikov (qKZ) connection on the trivial vector bundle over $\mathbb{C}^n$ with fiber $L_1 \otimes \ldots \otimes L_n$.

Consider the quasiclassical limit of the qKZ. Namely, set $y_i = z_i/h$, for some $h \in \mathbb{C}$ and let $h \to 0$. In this limit the qKZ equation turns into a system of differential equations

$$p\partial_{y_i} \tilde{\Psi}(y) = -\sum_{j \neq i} \frac{\tilde{\Omega}_{ij}}{y_i - y_j} \tilde{\Psi}(y), \qquad i = 1, \ldots, n, \tag{3}$$

where $\tilde{\Omega}_{ij} = \Omega_{ij} + A_{ij}$ and $A_{ij} \in \mathbb{C}$ is a constant defined by

$$\Omega v_i \otimes v_j = -A_{ij} v_i \otimes v_j,$$

v_i, v_j are highest weight vectors generating L_i, L_j, see Section 7 in [TV] and Section 12.5 in [CP].

Notice that if $p = -\kappa$, then $\Psi(z)$ is a solution of the KZ equation (2) if and only if the function

$$\tilde{\Psi}(y) = \prod_{i<j} (y_i - y_j)^{A_{ij}/\kappa}\, \Psi(y)$$

is a solution of the equation (3).

In this paper we suggest a quantization of the space of conformal blocks. Namely, under the resonance condition,

$$(\theta, \lambda) + p + N + k - 1 = 0, \tag{4}$$

where $k \in \mathbb{N}$, we introduce the space of quantized conformal blocks, $C_{\lambda_{n+1}}(z)$, by

$$C_{\lambda_{n+1}}(z) = \{m \in (L_1 \otimes \ldots \otimes L_n)^{sing}_{\lambda} \mid (e(z))^k m = 0\},$$

where

$$e(z)m = \sum_{j=1}^{n} \left(z_j - e^{(j)}_{NN} + \sum_{s=j+1}^{n} 2h^{(s)}_{\theta} \right) e^{(j)}_{\theta} m + \sum_{j=2}^{N-1} \sum_{\substack{r,s=1 \\ r<s}}^{N} e^{(r)}_{Nj} e^{(s)}_{j1} m,$$

and $e^{(s)}_{ij}$ denotes the element $e_{ij} \in gl(N)$ acting on the s-th factor of $L_1 \otimes \ldots \otimes L_n$.

The operator $e(z)$ can be described in terms of the action of the Yangian $Y(gl(N))$ in the tensor product of evaluation modules $L_1(z_1) \otimes \ldots \otimes L_n(z_n)$, $e(z) = T^{(2)}_{N1} - T^{(1)}_{NN} T^{(1)}_{N1}$, see (11).

In the quasiclassical limit the resonance condition (4) coincides with the resonance condition (1) and the operator $e(z)$ tends to the operator $E(z)$.

In this paper we show that the space of quantized conformal blocks is invariant with respect to the quantized Knizhnik-Zamolodchikov connection. The proof is based on an identity in the algebra with two generators x, y and defining relation $xy = yx + yy$.

In a recent paper [EF], B.Enriquez and G.Felder gave a construction of the space of quantized conformal blocks as a space of suitable coinvariants of an action of quantum doubles of Yangians. We expect that the Enriquez-Felder construction applied to our situation will identify the space of coinvariants with the space of conformal blocks introduced in this paper.

The simplest qKZ equation is the qKZ equation associated with $sl(2)$. In [TV], [MV] solutions of the $sl(2)$ qKZ equation were constructed in terms of multidimensional hypergeometric functions. In the next paper we will show that all hypergeometric solutions of the $sl(2)$ qKZ equation automatically belong to the space of quantized conformal blocks. This result is analogues to the fact that the values of all hypergeometric solutions of the KZ differential equation automatically belong to the space of conformal blocks of CFT, see [FSV1-3].

2. The qKZ connection

2.1. **The Lie algebra** $gl(N)$**.** Let N be a natural number. The Lie algebra $gl(N)$ is spanned over $\mathbb{C}$ by elements e_{ij}, $i, j = 1, \ldots, N$, with commutators given by

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}, \qquad i, j, k, l = 1, \ldots, N,$$

where δ_{ij} is the Kronecker symbol.

Let $\mathfrak{h}$ be Cartan subalgebra spanned by $h_i = \frac{1}{2} e_{i,i}$, $i = 1, \ldots, N$. Let θ be the highest root and let $h_\theta = h_1 - h_N$, $e_\theta = e_{1,N}$.

Let $\lambda = (\lambda_{(1)}, \dots, \lambda_{(N)}) \in \mathbb{C}^N$. Let V be the Verma module over $gl(N)$ with highest weight λ, V, i.e V is generated by a highest vector v such that $h_i v = \lambda_{(i)} v$ and $e_{ij} v = 0$ for $i, j = 1, \dots, N$, $i < j$.

Let $S \in V$ be the maximal proper submodule. Then $L = V/S$ is the irreducible $gl(N)$ module with highest weight λ.

For a $gl(N)$ module M with highest weight $\lambda = (\lambda_{(1)}, \dots, \lambda_{(N)}) \in \mathbb{C}^N$, let $M^{sing} \in M$ be the subspace of singular vectors, i.e. the subspace of vectors annihilated by $e_{i,i+1}$, for all $i = 1, \dots, N-1$,

$$M^{sing} = \{m \in M \mid e_{i,i+1} m = 0,\ i = 1, \dots, N-1\}.$$

Let also

$$(M)_l = \{m \in M \mid h_\theta m = (\lambda_{(1)} - \lambda_{(N)} - l)\, m\},$$

and $(M)_l^{sing} = (M)_l \bigcap M^{sing}$.

2.2. **The Hopf algebra $Y(gl(N))$.** The Yangian $Y(gl(N))$ is an associative algebra with an infinite set of generators $T_{i,j}^{(s)}$, $i, j = 1, \dots, N$, $s = 0, 1, \dots,$ subject to the following relations:

$$[T_{ij}^{(r)}, T_{kl}^{(s+1)}] - [T_{ij}^{(r+1)}, T_{kl}^{(s)}] = T_{kj}^{(r)} T_{il}^{(s)} - T_{kj}^{(s)} T_{il}^{(r)}, \qquad T_{ij}^{(0)} = \delta_{ij},$$

$i, j, k, l = 1, \dots, N$; $r, s, = 1, 2, \dots$.

The comultiplication $\Delta : Y(gl(N)) \to Y(gl(N)) \otimes Y(gl(N))$ is given by

$$\Delta : T_{ij}^{(s)} \mapsto \sum_{k=1}^{N} \sum_{r=0}^{s} T_{ik}^{(r)} \otimes T_{kj}^{(s-r)}.$$

For each $x \in \mathbb{C}$, there is an automorpfism $\rho_x : Y(gl(N)) \to Y(gl(N))$ given by

$$\rho_x : T_{ij}^{(s)} \mapsto \sum_{r=1}^{s} \binom{s-1}{r-1} x^{s-r} T_{ij}^{(r)}.$$

There is also an *evaluation morphism* ϵ to the universal enveloping algebra of $gl(N)$, $\epsilon : Y(gl(N)) \to U(gl(N))$, given by

$$\epsilon : T_{ij}^{s} \mapsto \delta_{1s} e_{ji},$$

for $s = 1, 2, \dots$.

Introduce the generating series $T_{ij}(u) = \sum_{s=0}^{\infty} T_{ij}^{(s)} u^{-s}$. In terms of these series the relations in the Yangian take the form

$$R(x-y) T_{(1)}(x) T_{(2)}(y) = T_{(2)}(y) T_{(1)}(x) R(x-y),$$

where $R(x) = (x \,\mathrm{Id} + P) \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$, $P \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ is the operator of permutation of the two factors, $T_{(1)}(x) = 1 \otimes T(x)$, $T_{(2)}(x) = T(x) \otimes 1$.

In terms of the generating series the comultiplication Δ, the automorphisms ρ_x and the evaluation morphism ϵ take the form

$$\Delta : T_{ij}(u) \mapsto \sum_{k=1}^{N} T_{ik}(u) \otimes T_{kj}(u),$$

$$\rho_x : T_{ij}(u) \mapsto T_{ij}(u-x),$$

$$\epsilon : T_{ij}(u) \mapsto \delta_{ij} + e_{ji} u^{-1},$$

$i, j = 1, \dots, N$. For more detail on the Yangian see [CP],[KR].

For any $gl(N)$ module M and $x \in \mathbb{C}$, let $M(x)$ be the $Y(gl(N))$ module obtained from the module M via the homomorphism $\epsilon \circ \rho(x)$. The module $M(x)$ is called the *evaluation module*. The action of $Y(gl(N))$ in the evaluation module $M(x)$ is given by

$$T_{ij}^{(s)} m = x^{s-1} e_{ji} m,$$

for all $m \in M$, $i, j = 1, \dots, N$, $s = 1, 2, \dots$.

Let L_1, L_2 be irreducible $gl(N)$ modules. For generic complex numbers x, y, the $Y(gl(N))$ modules $L_1(x) \otimes L_2(y)$ and $L_2(y) \otimes L_1(x)$ are irreducible and isomorphic. There is a unique intertwiner of the form $PR_{L_1L_2}(x-y)$ mapping $v_1 \otimes v_2$ to $v_2 \otimes v_1$, where P is the operator of permutation of the two factors and v_i are highest weight vectors generating L_i, $i = 1, 2$. The operator $R_{L_1L_2}(x) \in \mathrm{End}(L_1 \otimes L_2)$ is called the *rational R-matrix*, see [CP], [D].

Let L_1, L_1, L_3 be irreducible $gl(N)$ modules. The rational R-matrix satisfies the *Yang-Baxter equation* in $\mathrm{End}(L_1 \otimes L_2 \otimes L_3)$:

$$R_{L_1L_2}(x-y) R_{L_1L_3}(x) R_{L_2L_3}(y) = R_{L_2L_3}(y) R_{L_1L_3}(x) R_{L_1L_2}(x-y), \tag{5}$$

the symmetry relation:

$$PR_{L_1L_2}(x) = R_{L_2L_1}(x) P \in \mathrm{End}(L_1 \otimes L_2), \tag{6}$$

and the inversion relation:

$$R_{L_1L_2}(x) R_{L_1L_2}(-x) = 1 \in \mathrm{End}(L_1 \otimes L_2). \tag{7}$$

For all $x \in \mathbb{C}$, the rational R-matrix $R_{L_1L_2}(x)$ commutes with the action of $gl(N)$ in $L_1 \otimes L_2$ and, in particular, preserves the weight decomposition.

2.3. **The qKZ connection.** Fix a non-zero complex number p. Let L_i, $i = 1, \dots, n$, be irreducible $gl(N)$ modules. For $m = 1, \dots, n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, define the *Knizhnik-Zamolodchikov operators* $K_m(z) \in \mathrm{End}(L_1 \otimes \dots \otimes L_n)$ by the formula

$$K_m(z) = R_{L_mL_{m-1}}(z_m - z_{m-1} + p) \dots R_{L_mL_1}(z_m - z_1 + p) \times \tag{8}$$

$$\times R_{L_mL_n}(z_m - z_n) \dots R_{L_mL_{m+1}}(z_m - z_{m+1}). \tag{9}$$

The KZ operators commute with the action of $gl(N)$, $K_m(z) e_{ij} = e_{ij} K_m(z)$ for all $z \in \mathbb{C}^n$, $i, j = 1, \dots, N$ and $m = 1, \dots, n$. In particular, the KZ operators preserve the subspaces $(L_1 \otimes \dots \otimes L_n)_l$ and $(L_1 \otimes \dots \otimes L_n)_l^{sing}$ for all $l \in \mathbb{Z}_{\geq 0}$.

Lemma 1. *The qKZ operators $K_m(z)$ and the rational R-matrices $R_{L_iL_{i+1}}(x)$ satisfy the following relations*

$$K_{i+1}(z_1, \dots, z_{i+1}, z_i, \dots, z_n) P_{L_iL_{i+1}} R_{L_iL_{i+1}}(z_i - z_{i+1}) = P_{L_iL_{i+1}} R_{L_iL_{i+1}}(z_i + p - z_{i+1}) K_i(z),$$

$$K_k(z_1, \dots, z_j + p, \dots, z_n) K_j(z_1, \dots, z_n) = K_j(z_1, \dots, z_k + p, \dots, z_n) K_k(z_1, \dots, z_n),$$

for all, $k, j = 1, \dots, n$ and $i = 1, \dots, n-1$.

Lemma 1 follows from properties (5)-(7) of the R-matrix.

The operators $K_m(z)$, $m = 1, \dots, n$, define a discrete flat connection on the trivial vector bundle over $\mathbb{C}^n$ with fiber $L_1 \otimes \dots \otimes L_n$. This connection is called the *quantized Knizhnik-Zamolodchikov connection*. For all $l \in \mathbb{Z}_{\geq 0}$, the subspaces $(L_1 \otimes \dots \otimes L_n)_l$ and $(L_1 \otimes \dots \otimes L_n)_l^{sing}$ are invariant under the qKZ connection.

A subspace $(L_1\otimes\ldots\otimes L_n)_l^{sing}$ is called a *resonance subspace* if for some $k\in\mathbb{N}$, $k\le l$,

$$2h_\theta+(p+N+k-1)\,\mathrm{Id}=0 \tag{10}$$

in $(L_1\otimes\ldots\otimes L_n)_l^{sing}$.

Set

$$e(z)=T_{N1}^{(2)}-T_{NN}^{(1)}T_{N1}^{(1)}\in Y(gl(N)). \tag{11}$$

For each $m\in L_1\otimes\ldots\otimes L_n$, we have

$$e(z)m=\sum_{j=1}^{n}\left(z_j-e_{NN}^{(j)}+\sum_{s=j+1}^{n}2h_\theta^{(s)}\right)e_\theta^{(j)}m+\sum_{j=2}^{N-1}\sum_{\substack{r,s=1\\ r<s}}^{N}e_{Nj}^{(r)}e_{j1}^{(s)}m, \tag{12}$$

where $e_{ij}^{(s)}$ denotes e_{ij} acting on the s-th factor of $L_1\otimes\ldots\otimes L_n$.

Introduce the *subspace of quantized conformal blocks* $C(z)\subseteq(L_1\otimes\ldots\otimes L_n)_l^{sing}$. For a resonance subspace $(L_1\otimes\ldots L_n)_l^{sing}$, let $C(z)$ be the kernel of the operator $(e(z))^k$ acting in $(L_1\otimes\ldots\otimes L_n)_l^{sing}$.

$$C(z)=\{m\in(L_1\otimes\ldots\otimes L_n)_l^{sing}\,|\,(e(z))^km=0\}.$$

For a non-resonance subspace $(L_1\otimes\ldots\otimes L_n)_l^{sing}$, let $C(z)=(L_1\otimes\ldots\otimes L_n)_l^{sing}$.

Theorem 2. *Let L_{λ_i}, $i=1,\ldots,n$, be irreducible $gl(N)$ modules. Then the space of conformal blocks $C(z)$ is invariant with respect to the qKZ connection,*

$$K_i(z)C(z)=C(z_1,\ldots,z_i+p,\ldots,z_n),$$

as well as with respect to permutations of variables,

$$PR_{L_iL_{i+1}}(z_i-z_{i+1})C(z)=C(z_1,\ldots,z_{i+1},z_i,\ldots,z_n).$$

Theorem 2 is proved in Section 2.5.

Remark. It follows from the proof that the supspace of conformal blocks $C(z)$ in a resonance subspace $(L_1\otimes\ldots\otimes L_n)_l^{sing}$ can be also defined as the kernel of the operator $(T_{N1}^{(2)})^k$ acting in $(L_1\otimes\ldots\otimes L_n)_l^{sing}$.

2.4. **The Jordan plane.** The proof of Theorem 2 is based on an identity which holds in the algebra called the Jordan plane.

The associative algebra with generators x,y subject to the relation $xy=yx+yy$ is called the *Jordan plane* and is denoted A^J, $A^J=\mathbb{C}\langle x,y\rangle/(xy-yx-yy)$, see [DMMZh].

Let $q\in\mathbb{C}$, $q\ne0$. The associative algebra with generators x,y subject to the relation $xy=qyx$ is called the *quantum plane with parameter* q and is denoted A_q, $A_q=\mathbb{C}\langle x,y\rangle/(xy=qyx)$. The quantum plane with parameter 1 is isomorphic to the ring of polynomials in commuting variables x,y, $A_1\simeq\mathbb{C}[x,y]$.

An associative algebra A is called *quadratic* if it has the form $A=\mathbb{C}\langle x_1,\ldots,x_s\rangle/(I)$, and the ideal I is generated by homogenious polynomials in $x_1,\ldots,x_s$ of degree two. We have $A=\bigoplus_{r=0}^{\infty}(A)_r$, where $(A)_r$ is the r-th homogenious component spanned over $\mathbb{C}$ by homogenious polynomials in $x_1,\ldots,x_r$ of degree r.

A quadratic algebra A with two generators has a *polynomial growth* if $\dim(A)_r=\dim\mathbb{C}[x,y]_r=r+1$.

Theorem 3. *Any quadratic algebra in two generators with a polynomial growth is isomorphic either to the Jordan plane A^J or to the quantum plane A_q with some parameter q. The Jordan plane and quantum planes with different parameters are non-isomorphic except for quantum planes with inverse parameters, $A_q \simeq A_{q^{-1}}$.*

Theorem 3 is proved by a direct computation.

Theorem 4. *Let x, y be generators of the Jordan plane A^J. For any complex number p and a natural number k, the following identity holds:*

$$(x+py)^k = (x+(p-k+1)y)(x+(p-k+3)y)(x+(p-k+5)y)\dots(x+(p+k-1)y).$$

Proof: For $l \in \mathbb{N}$, we have $xy^l = y^l + ly^{l+1}$. By induction on k, we prove that both the right and left hand sides in formula (4) are equal to

$$\sum_{i=0}^{k} \binom{k}{i} p(p+1)\dots(p+i-1)y^i x^{k-i}.$$

□

2.5. **Proof of Theorem 2.** Let $(L_1 \otimes \dots L_n)_l^{sing}$ be a resonance subspace, otherwise the Theorem is trivial. We prove that for each $m \in (L_1 \otimes \dots L_n)_l^{sing}$,

$$K_i(z)(e(z_1,\dots,z_n))^k m = (e(z_1,\dots,z_i+p,\dots,z_n))^k K_i(z)m, \tag{13}$$

for $i = 1,\dots,n$.

For $j = 1,\dots,n$, we have

$$PR_{L_jL_{j+1}}(z_j - z_{j+1})(e(z))^k = (e(z))^k PR_{L_jL_{j+1}}(z_j - z_{j+1}), \tag{14}$$

since $e(z) \in Y(gl(2))$ and $PR_{L_jL_{j+1}}(z_j - z_{j+1})$ is an intertwiner of Yangian modules. Therefore, the subspace of conformal blocks is invariant with respect to permutations of variables, $PR_{L_iL_{i+1}}(z_i - z_{i+1})C(z) = C(z_1,\dots,z_{i+1},z_i,\dots,z_n)$.

Consider the case $i = 1$. Using (14), we get

$$K_1(z)(e(z))^k = ((P_{2,\dots,n,1})^{-1}e(z)P_{2,\dots,n,1})^k K_1(z),$$

where $P_{2,\dots,n,1}$ is the permutation operator, $P_{2,\dots,n,1}\, m_1 \otimes \dots \otimes m_n = m_2 \otimes \dots \otimes m_n \otimes m_1$, for $m_i \in L_i$, $i = 1,\dots,n$.

Since $K_1(z)$ preserves $(L_1 \otimes \dots \otimes L_n)_l^{sing}$, it suffices to show that for each $m \in (L_1 \otimes \dots \otimes L_n)_l^{sing}$,

$$((P_{2,\dots,n,1})^{-1}e(z)P_{2,\dots,n,1})^k m = (e(z_1+p,z_2,\dots,z_n))^k m.$$

We have

$$e(z_1+p,z_2,\dots,z_n) = e(z) + p\, e_\theta^{(1)}.$$

For $m \in L_1 \otimes \dots \otimes L_n$, from (12), we get

$$((P_{2,\dots,n,1})^{-1}e(z)P_{2,\dots,n,1})^k m = \tag{15}$$

$$\left(e(z) + 2h_\theta^{(1)}e_\theta - (2h_\theta + (N-2))e_\theta^{(1)} + \sum_{j=2}^{N-2} (e_{j1}^{(1)}e_{Nj} - e_{Nj}^{(1)}e_{j1}) \right)^k m.$$

For $s = 1, \ldots, n$, we have the commutation relations

$$e_{Nj}^{(s)} e(z) = e(z) e_{Nj}^{(s)} - e_\theta^{(s)} e_{Nj}^{(s)} - \sum_{r=1}^{s-1} 2 e_\theta^{(r)} e_{Nj}^{(s)}, \tag{16}$$

$$e_{Nj} e(z) = e(z) e_{Nj} - e_\theta e_{Nj},$$

where $j = 1, \ldots, N-1$. Similarly,

$$e_{j1}^{(s)} e(z) = e(z) e_{j1}^{(s)} - e_\theta^{(s)} e_{j1}^{(s)} - \sum_{r=1}^{s-1} 2 e_\theta^{(r)} e_{j1}^{(s)},$$

$$e_{j1} e(z) = e(z) e_{j1} - e_\theta e_{j1},$$

for $j = 2, \ldots, N$.

Now, for any $m \in (L_1 \otimes \ldots \otimes L_n)^{sing}$, we use the commutation relations to move e_{Nj}, e_{j1} and e_θ in formula (15) to the right and get

$$((P_{2,\ldots,n,1})^{-1} e(z) P_{2,\ldots,n,1})^k = (e(z) - (2h_\theta + (N-2)) e_\theta^{(1)})^k \, m,$$

since $e_{Nj} m = e_{j1} m = e_\theta m = 0$ for $j = 2, \ldots, N-1$.

For $s \in \mathbb{Z}_{\geq 0}$, $m \in (L_1 \otimes \ldots \otimes L_n)_s$, we have $e(z)m, e_\theta^{(1)} m \in (L_1 \otimes \ldots \otimes L_n)_{s-1}$. Therefore, for each $m \in (L_1 \otimes \ldots \otimes L_n)_l^{sing}$, the resonance condition (10) implies

$$\begin{aligned} &(e(z) - (2h_\theta + (N-2)) e_\theta^{(1)})^k \, m = \\ &\quad (e(z) + (p-k+1) e_\theta^{(1)}) \ldots (e(z) + (p+k-3) e_\theta^{(1)})(e(z) + (p+k-1) e_\theta^{(1)}) \, m. \end{aligned}$$

Notice that the operators $e(z)$ and $e_\theta^{(1)}$ define a representation of the Jordan algebra, $e(z) e_\theta^{(1)} = e_\theta^{(1)} e(z) + e_\theta^{(1)} e_\theta^{(1)}$, see formula (16) for $j = s = 1$.

For $i = 1$, formula (13) follows from Theorem 4.

For $i = 2, \ldots, n$, formula (13) follows from formula (13) for $i = 1$, formula (14) and the first formula in Lemma 1. $\square$

Notice that $gl(N)$ can be replaced with $sl(N)$. Namely, we consider a tensor product of evaluation modules over $Y(sl(N))$. The same proof shows that the subspace of conformal blocks defined as in Section 2.3 is invariant under the qKZ connection.

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